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# Note

# A Remark on the Application of Closed and Semi-closed Quadrature Rules to the Direct Numerical Solution of Singular Integral Equations

The direct quadrature method of numerical solution of singular integral equations with Cauchy-type kernels is modified so as to become applicable to the cases when closed and semi-closed quadrature rules are used, the index of the integral equation is equal to zero, and the number of collocation points is less, by one, than the number of nodes used in the quadrature rules. The proposed modification consists in using a theoretically determined condition, not involving principal value integrals. Numerical applications are also made.

## 1. Preliminary Considerations

The direct quadrature method of numerical solution of singular integral equations with Cauchy-type kernels (called simply singular integral equations in what follows) consists in the application of a quadrature rule for the approximation to the integrals of the singular integral equation followed by the reduction of the resulting approximate equation to a system of linear equations by applying this equation at a set of appropriately selected collocation points. Erdogan *et al.* [1] considered the solution of singular integral equations of the first kind by a direct method, whereas Krenk [2] generalized the results of Ref. [1] to the case of singular integral equations of the second kind but with constant coefficients. Further results on this method were obtained by Ioakimidis [3] and reported in a series of papers by Ioakimidis and Theocaris, the most general of which is Ref. [4].

Here we will confine ourselves to singular integral equations of the first kind of the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{t-x}\,dt+\int_{-1}^{1}k(t,x)\varphi(t)\,dt=f(x),\qquad -1<x<1, \tag{1}$$

where $k(t,x)$ and $f(x)$ are known functions, whereas $\varphi(t)$ is the unknown function to be determined. It is well known that the unknown function $\varphi(t)$ behaves like [1, 5]

$$\varphi(t)=\omega(t)g(t), \tag{2}$$



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where $\omega(t)$ is a weight function of the form

$$\omega(t) = (1-t)^{\alpha}(1+t)^{\beta}, \qquad \alpha = \pm\tfrac{1}{2}, \quad \beta = \pm\tfrac{1}{2} \tag{3}$$

and $g(t)$ is a regular function if $k(t, x)$ and $f(x)$ are assumed also to be regular functions. For the numerical solution of Eq. (1) we can use an appropriate numerical integration rule of the form

$$\int_{-1}^{1} \omega(t) g(t)\, dt \simeq \sum_{i=1}^{n} A_i g(t_i) \tag{4}$$

with $n$ nodes $t_i$ and weights $A_i$, and reduce Eq. (1) to the system of linear algebraic equations

$$\sum_{i=1}^{n} A_i \left[\frac{1}{\pi(t_i - x_k)} + k(t_i, x_k)\right] \tilde{g}(t_i) = f(x_k), \qquad k = 1(1)(n-\kappa), \tag{5}$$

where $\tilde{g}(t_i)$ are the approximate values of $g(t)$ at the nodes $t_i$ used,

$$\kappa = -(\alpha + \beta) \tag{6}$$

is the index of Eq. (1) [5], restricted in practical problems to the values $\kappa = 1$ and $\kappa = 0$, and $x_k$ are appropriate collocation points selected in such a way that Eq. (4) holds true for Cauchy-type principal value integrals too [1–4].

From Eqs. (5) we observe that, although in the case when $\kappa = 0$ the number of these equations is equal to the number of unknowns $\tilde{g}(t_i)$, yet this does not hold true if $\kappa = 1$. In this case, a collocation point is missing. But the theoretical results of Muskhelishvili [5] show that in this case Eq. (1) does not have a unique solution unless supplemented by an additional condition of the form

$$\int_{-1}^{1} \varphi(t)\, dt = \int_{-1}^{1} \omega(t) g(t)\, dt = C, \tag{7}$$

where $C$ is a known constant. Then the application of Eq. (4) to Eq. (7) yields

$$\sum_{i=1}^{n} A_i \tilde{g}(t_i) = C \tag{8}$$

and Eqs. (5) and (8) are equal in number to the unknowns $\tilde{g}(t_i)$ $(i = 1(1)n)$.

This ideal situation disappears if Eq. (4) is a semi-closed or closed quadrature rule; that is, a rule in which one or two nodes $t_i$ ($i = 1$ and/or $i = n$) coincide with the end-points $t = \pm 1$ of the integration interval $[-1, 1]$. Such rules were applied to the numerical solution of singular integral equations for the first time by Ioakimidis [3] and considered in detail in Refs. [7–10]. The main reason favoring the use of these rules is that, in this way, the determination of the values of the unknown function $g(t)$ at the end-points $t = \pm 1$ of the integration interval $[-1, 1]$ is achieved directly from the solution of Eqs. (5) (supplemented by Eq. (8) if $\kappa = 1$) without using extrapolation techniques. These values are of particular interest in some physical problems. In fact, Eq. (1) appears in several physical problems, like solid mechanics problems (crack problems, contact problems, dislocations problems, etc.) [1, 3], as well as fluid mechanics problems (problems of hydrodynamics and aerodynamics, etc.) [6, 12, 13]. In fluid mechanics problems, the collocation method (permitting the determination of $\tilde{g}(\pm 1)$) based on Jacobi polynomials has been generally used [6] instead of the quadrature method [1–4, 7–10], which is used here too.

Unfortunately, the results of Ref. [8] make it clear that the number of collocation points is equal to $(n-1)$ if we use Lobatto–Jacobi quadrature rules for singular integral equations. Of course, if $\kappa = 1$, then Eq. (8) supplements Eqs. (5) and a system of $n$ linear algebraic equations results. But if $\kappa = 0$, as happens in a series of physical problems (like most problems of flow of fluids [6, 12, 13] or problems associated with dislocations arrays or even contact problems in the theory of elasticity [1, 3]), then, evidently, Eq. (1) cannot be solved by using the Lobatto–Jacobi quadrature rules. Similar results also hold true in several cases (but not always) when using Radau–Jacobi quadrature rules [8] associated with the weight function $\omega(t)$. It can be said that closed quadrature rules generally provide only $(n-1)$ collocation points, whereas semi-closed quadrature rules provide $(n-1)$ collocation points (when $\kappa = 0$) in about half of the cases.

Up to now no remedy has been found for this unfortunate situation when $\kappa = 0$. Krenk [10] suggested that $\alpha$ and $\beta$ in Eqs. (3) be always restricted to taking negative values (that is reduced by 1 if they result positive from the physical problem under consideration). In this case, the index $\kappa$ results from Eq. (6) in being equal to 1 and the additional condition

$$g(1) = 0 \qquad \text{or} \qquad g(-1) = 0 \tag{9}$$

is now available and substitutes Eq. (7). But, obviously, Eqs. (9) say that the node $t = 1$ or $t = -1$ does not exist anymore in Eq. (4) since we have replaced $g(t)$ by $(1-t)g(t)$ or $(1+t)g(t)$ when reducing $\alpha$ or $\beta$ by 1 and using one of Eqs. (9). Hence, the values $g(1)$ or $g(-1)$ of the original function $g(t)$ will not be determined by this procedure and the conditions (9) for the new functions having replaced $g(t)$. Here we will propose a new technique for the solution of Eq. (1) in all cases by using closed or semi-closed quadrature rules with $\kappa = 0$.

## 2. The Proposed Technique

We will illustrate the proposed technique in the case when $\alpha = +\frac{1}{2}$ and $\beta = -\frac{1}{2}$ in Eqs. (3). Then the index $\kappa$ of Eq. (1) results from Eq. (6) in being equal to 0. Following the previous developments, we assume $\alpha$ is replaced by

$$\gamma = \alpha - 1 = -\tfrac{1}{2} \tag{10}$$

and $\omega(t)$ by

$$w(t) = (1-t)^{\gamma}(1+t)^{\beta} = (1-t)^{-1}\omega(t) \tag{11}$$

with $\omega(t)$ given by Eq. (3). Then Eq. (2) will take the form

$$\varphi(t) = w(t)h(t), \tag{12}$$

where $h(t)$ is a new unknown function, related to $g(t)$ by

$$h(t) = (1-t)g(t) \tag{13}$$

as is clear from Eqs. (2), (11) and (12). Evidently, $h(t)$ satisfies the condition

$$h(1) = 0. \tag{14}$$

Now we take into account that Eq. (1) is equivalent to the following Fredholm integral equation of the second kind [5, 6]:

$$\begin{aligned} h(t) + \frac{1}{\pi}\int_{-1}^{1} w(y)\left[\int_{-1}^{1} \frac{1}{w(x)}\frac{k(y,x)}{t-x}\,dx\right] h(y)\,dy \\ = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{w(x)}\frac{f(x)}{t-x}\,dx + \frac{C}{\pi}, \qquad -1 \leqslant t \leqslant 1, \end{aligned} \tag{15}$$

where the constant $C$ is related to $h(t)$ by the condition

$$\int_{-1}^{1} w(t)h(t)\,dt = C, \tag{16}$$

analogous to Eq. (7). To satisfy Eq. (14), we apply Eq. (15) at $t = 1$ and we find

$$\int_{-1}^{1} \omega(t)\left[\int_{-1}^{1} \frac{k(t,x)}{\omega(x)}\,dx - 1\right] g(t)\,dt = \int_{-1}^{1} \frac{f(x)}{\omega(x)}\,dx, \tag{17}$$

taking also into account Eqs. (11), (13), (14) and (16). This is a condition for the original function $g(t)$ and, like Eq. (7), it does not contain Cauchy-type principal value integrals. This condition provides the last linear algebraic equation to supplement Eqs. (5) if $\kappa = 0$ and a collocation point is missing due to the application of a closed or, sometimes, a semi-closed quadrature rule of the form (4).

If we put

$$K(t) = \int_{-1}^{1} \frac{k(t,x)}{\omega(x)}\,dx - 1, \qquad F = \int_{-1}^{1} \frac{f(x)}{\omega(x)}\,dx, \tag{18}$$

and apply Eq. (4) to Eq. (17), we obtain

$$\sum_{i=1}^{n} A_i K(t_i) \tilde{g}(t_i) = F. \tag{19}$$

Equation (19) is analogous to Eq. (8) obtained in the case when $\kappa = 1$. Finally, as regards the evaluation of $K(t)$ and $F$, this can easily be achieved by using an appropriate numerical integration rule with a sufficiently large number of nodes $m$ [11].

It can also be mentioned that quite similar results are obtained if we assume that $\alpha = -\frac{1}{2}$ and $\beta = +\frac{1}{2}$ (so that $\kappa = 0$ again) in Eq. (1). The generalization of the present results to singular integral equations of the second kind [2, 8, 10] is trivial on the basis of the above developments.

## 3. Numerical Applications

As an application we consider again Eq. (1) with $\omega(t)$ given by Eq. (3) with $\alpha = +\frac{1}{2}$ and $\beta = -\frac{1}{2}$ and we will use for its numerical solution the device proposed in the previous section. The nodes $t_i$ and the weights $A_i$ in Eq. (4) (with $\alpha = +\frac{1}{2}$ and $\beta = -\frac{1}{2}$ as previously) are those used in the classical Lobatto–Jacobi quadrature rule and reported by Kopal [11]. On the other hand, the $(n-1)$ collocation points $x_k$ are the roots of the Jacobi polynomial $P_n^{(-\alpha-1,-\beta-1)}(x) \equiv P_n^{(-3/2,-1/2)}(x)$ [8]. The computer programs already used in Ref. [8] were used once more for the determination of the collocation points $x_k$, the nodes $t_i$ and the weights $A_i$. Their values for $n = 7$ are presented in Table I and we clearly observe the fact that the $(n-1)$

TABLE I

Collocation Points $x_k$, Nodes $t_i$ and Weights $A_i$ for the Lobatto–Jacobi Method with $\alpha = +\frac{1}{2}$, $\beta = -\frac{1}{2}$ and $n = 7$

| $x_k$ | $t_i$ | $A_i$ |
|---|---|---|
| | 1.00000 | $0.86307 \times 10^{-2}$ |
| 0.90839 | | |
| | 0.76954 | 0.11714 |
| 0.58686 | | |
| | 0.37081 | 0.30985 |
| 0.13393 | | |
| | −0.11003 | 0.54225 |
| −0.34690 | | |
| | −0.56291 | 0.76114 |
| −0.74551 | | |
| | −0.88409 | 0.91638 |
| −0.97059 | | |
| | −1.00000 | 0.48620 |

TABLE II

Convergence of the Numerical Results $\tilde{g}_2(\pm 1)$ for the Solution $g(t)$ of Eq. (1) with $\alpha = +\frac{1}{2}$ and $\beta = -\frac{1}{2}$ at the Nodes $t = \pm 1$[a]

| | $f(x)$: $-1$ | $-\exp x$ | | $-1$ | |
|---|---|---|---|---|---|
| | $k(t, x)$: 0 | 0 | | $\frac{1}{2.5}\cot\frac{\pi(t-x)}{2.5} - \frac{1}{\pi(t-x)}$ | |
| $n$ | $\tilde{g}(-1) = \tilde{g}(+1)$ | $\tilde{g}_2(-1)$ | $\tilde{g}_2(+1)$ | $\tilde{g}_2(-1)$ | $\tilde{g}_2(+1)$ |
| 3 | 1.0000 | 1.2666 | 4.9231 | 1.4895 | 6.4238 |
| 4 | 1.0000 | 1.2661 | 4.9285 | 1.5569 | 6.6645 |
| 5 | 1.0000 | 1.2661 | 4.9285 | 1.5640 | 6.6881 |
| 6 | 1.0000 | 1.2661 | 4.9285 | 1.5648 | 6.6911 |
| 7 | 1.0000 | 1.2661 | 4.9285 | 1.5650 | 6.6915 |

[a] Obtained by using the Lobatto–Jacobi method in its modified form proposed in this paper.

collocation points $x_k$ alternate with the $n$ nodes $t_i$ used in Eq. (4). We can also mention that for the numerical evaluation of the integrals in Eqs. (18) we used also a Lobatto–Jacobi quadrature rule with the same number of nodes, but for the weight function $1/\omega(x)$.

In Table II we present the numerical results obtained by the above-described method of numerical solution of Eq. (1) for three selections of the pair of functions $k(t, x)$ and $f(x)$. We observe from the numerical results of Table II that they converge rapidly for increasing values of $n$. The "airfoil equation" (1), for $f(x) \equiv -1$ and $k(t, x) \equiv 0$, possesses the closed-form solution $g(t) = 1$ [13] recovered from the numerical results of Table II. This is the problem of flow of an ideal fluid past a straight segment [13]. Similarly, the third case considered in Table II corresponds to a periodic array of straight segments along a straight line with a ratio of the period $b$ of the array to the length $a$ of the straight segments equal to $b/a = 1.25$.

## 4. An Alternative Technique

Instead of using the technique proposed in Section 2 in the case when one collocation point is missing and the system of equations (5) is a system of $(n-1)$ equations with $n$ unknowns (with $\kappa = 0$) when a closed or, sometimes, a semi-closed quadrature rule of the form (4) is used, we can, alternatively, use as an additional collocation point $x_n$ the end-point $t = 1$ or $t = -1$ of the integration interval $[-1, 1]$ corresponding to the positive exponent $\alpha$ or $\beta$ in Eqs. (3). This is clearly possible for this end-point and the integrals in Eq. (1) are regular integrals when using this end-point as a collocation point. Of course, it is not permissible to use the other end-point of the integration interval as a collocation point since the first integral in Eq. (1) does not exist in this case.

The only disadvantage of this technique is that Eq. (4) is not appropriate anymore for the evaluation of the first integral in Eq. (1) since the weight function for this integral has now changed. A new quadrature formula, to replace (4) only for the first integral of Eq. (1) for the new weight function but with the same set of nodes $t_i$, has to be constructed if the technique of this section is preferred. Yet, generally, this is not easily possible since such a formula includes the derivative of the integrand at the end-point under consideration, which is an additional unknown. This derivative has to be eliminated by interpolation, but this reduces considerably the accuracy of the quadrature rule.

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## REFERENCES

1. F. ERDOGAN, G. D. GUPTA, AND T. S. COOK, *in* "Mechanics of Fracture," Vol. 1, "Methods of Analysis and Solutions of Crack Problems," (G. C. Sih, Ed.), Chap. 7, pp. 368–425, Noordhoff, Leyden, The Netherlands, 1973.
2. S. KRENK, *Quart. Appl. Math.* **33** (1975), 225–232.
3. N. I. IOAKIMIDIS, "General Methods for the Solution of Crack Problems in the Theory of Plane Elasticity," Doctoral thesis, The National Technical University of Athens, Athens, 1976 (University Microfilms Order No. 76-21,056).
4. P. S. THEOCARIS AND N. I. IOAKIMIDIS, *J. Comput. Phys.* **30** (1979), 309–323.
5. N. I. MUSKHELISHVILI, "Singular Integral Equations," Noordhoff, Groningen, The Netherlands, 1953.
6. M. A. GOLBERG (Ed.), "Solution Methods for Integral Equations," Plenum, New York, 1979.
7. P. S. THEOCARIS AND N. I. IOAKIMIDIS, *Quart. Appl. Math.* **35** (1977), 173–183.
8. N. I. IOAKIMIDIS AND P. S. THEOCARIS, *J. Math. Phys. Sci.* **11** (1977), 219–235.
9. N. I. IOAKIMIDIS AND P. S. THEOCARIS, *Apl. Mat.* **23** (1978), 439–452.
10. S. KRENK, *J. Inst. Math. Its Appl.* **22** (1978), 99–107.
11. Z. KOPAL, "Numerical Analysis," 2nd ed., Chapman & Hall, London, 1961.
12. F. G. TRICOMI, "Integral Equations," Interscience, New York, 1957.
13. S. G. MIKHLIN, "Integral Equations," Pergamon, Oxford, 1957.

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NIKOLAOS I. IOAKIMIDIS*

*Chair of Mathematics B′,*
*School of Engineering*
*University of Patras*
*P.O. Box 25 B*
*Patras, Greece*

* Associate Professor of Mathematics at the School of Engineering of the University of Patras, Greece.